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A transformation of elastic boundary value problems with application to anisotropic behavior

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Abstract

A general geometrical transformation of the coordinates and of the displacement field is proposed; it is used to convert any boundary value problem for a linear elastic body into another one with different geometry, elastic moduli and boundary conditions. With this method, new problems, especially for inhomogeneous anisotropic bodies, may be solved by use of solutions of simpler ones. After a derivation of sufficient conditions to be fulfilled by such a transformation, the case of a linear homogeneous transformation is investigated in more detail. It is shown that a number of situations exist for which the transformed problem has a known analytical solution which can be used to derive the solution of the original problem straightforwardly. Special attention is paid to Saint-Venant-type anisotropy and to the derivation of the Green function for an infinite or a semi-infinite body.

Keywords: Linear elasticity; Anisotropy; Transformation; Green functions; Saint-Venant anisotropy

1. Introduction

Methods for transforming complex problems into simpler ones have been frequently developed in various fields of mechanics and physics. For linear elasticity, which is considered here, most of them rely on a transformation of the coordinates, either linear or nonlinear. More recently, the simultaneous transformation of mechanical variables, such as the displacement field, has been also proposed.

Linear transformations of the coordinates have been used to convert the physical study of the response of anisotropic bodies into the resolution of isotropic problems. For thermal, hydraulic or chemical

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diffusion, a flux is created by the gradient of some scalar potential and the local constitutive equations involve a second-order conductivity tensor. For example, this method has been intensively used for the investigation of the hydraulic diffusion in media with an anisotropic permeability (Castany, 1967; Magnan, 2000). Anisotropic elasticity problems, which make use of a fourth-order tensor for the moduli or the compliances, are more complex and cannot be converted by the same method into problems of elastic isotropy in the general case (see below). Nevertheless, specific situations, such as plane problems, can be addressed and simplified with respect to the geometry or to the anisotropy (Green and Zerna, 1954; Alphutova et al., 1995): for instance, Green and Zerna have used complex variables to define stress potentials for an anisotropic infinite body with a symmetry plane and a circular hole loaded on its boundary and, thanks to a change of variables in the complex plane, to extend this solution to the case of an elliptic hole.

Nonlinear coordinate transformations have also been considered for the resolution of some elasticity problems. Lekhnitskii (1963) has addressed the torsion problem for a symmetric body around its symmetry axis and shown that, for some types of anisotropy, the stress potentials can be derived from that found for the isotropic case. The used transformation is linear with respect to the cylindric coordinates r and z and then nonlinear with respect to cartesian coordinates. Neither Green–Zerna's nor Lekhnitskii's transformations can be extended directly to general three-dimensional situations. The same comment holds for the conformal transformation which is used in perfect fluid mechanics in association with the complex potential method.

A linear transformation of both the coordinates and the displacement field which leads to the modification of the geometry and the elastic anisotropy has also been used for the investigation of Eshelby's inclusion problem or for the prediction of the response of heterogeneous elastic media. With this method, Milgrom and Shtrikman (1992) have generalized some results related to the elastic energy of the inclusion and suggested that the Eshelby tensor could also be calculated. Milton (2002) succeeded in generalizing some homogenization results for thermal properties; for instance, an adequate transformation of both the coordinates and the flux can lead to a new problem with isotropic behavior. Nevertheless these contributions, which are restricted to zero volume forces and tractions on the boundary, do not derive from any more general systematic transformation method.

A more systematic approach has been proposed by Pouya (2000). With help of a linear transformation of the coordinates and of the displacement field, the boundary value problem of an elastic body with arbitrary geometry and regular boundary conditions is converted into another one, with different geometry and anisotropy. This transformation has then been used, independently of the above mentioned methods, to extend classical solutions for the Eshelby and for the Green isotropic problems to specific classes of anisotropy; it has also been applied by Pouya and Reiffsteck (2003) to the resolution of the problem of foundations in anisotropic elastic soils. Nevertheless it is restricted to homogeneous media since it derives from a linear transformation of the coordinates.

In this paper, we propose an extension of this approach by considering a nonlinear transformation of the coordinates and of the displacement field. The conditions for which the transformed boundary value problem still refers to an anisotropic elastic body are derived and discussed. Sufficient conditions are exhibited which contain the Pouya (2000) transformation as a special case; this case is then investigated in more detail with application to several problems, including the derivation of Green functions, for Saint-Venant-type anisotropy.

2. Notation

In what follows, light-face (Greek or Latin) letters denote scalars; underlined letters designate vectors and bold-face letters, second-order tensors; outline letters are reserved for fourth-order tensors. The convention of summation on repeated indices is used implicitly.

The tensor product of two vectors is labelled as $\underline{a} \otimes \underline{b}$ and defined as follows for cartesian coordinates: $(\underline{a} \otimes \underline{b})_{ij} = a_i b_j$. The inner product of two vectors is labelled as $\underline{a} \cdot \underline{b} = a_i b_i$, the inner product of two second-order tensors as $\boldsymbol{a} : \boldsymbol{b} = a_{ij} b_{ij}$, the product of two second-order tensors by $\boldsymbol{a} \cdot \boldsymbol{b}$ with $(\boldsymbol{a} \cdot \boldsymbol{b})_{ij} = a_{ik} b_{kj}$. The operation of a second-order tensor $\boldsymbol{a}$ on a vector $\underline{n}$ is labelled as $\boldsymbol{a} \cdot \underline{n}$, $(\boldsymbol{a} \cdot \underline{n})_i = a_{ij} n_j$; when a fourth-order tensor $\mathbb{C}$ is acting on a second-order tensor $\boldsymbol{a}$, one has $(\mathbb{C} : \boldsymbol{a})_{ij} = C_{ijkl} a_{kl}$. The Euclidian norm is labelled as $\|\cdot\|$ with $\|\mathbb{C}\| = \sqrt{C_{ijkl} C_{ijkl}}$. The tensor transposed from $\boldsymbol{a}$ is denoted $\boldsymbol{a}^{\mathrm{T}}$.

The completely antisymmetric Levi–Civita tensor is denoted ϵ_{ijk} with the components:

$\epsilon_{ijk} = 1$ if i, j, k is an even permutation of $1,2,3$,
$\epsilon_{ijk} = -1$ if i, j, k is an odd permutation of $1,2,3$,
$\epsilon_{ijk} = 0$ otherwise.

The external product of two vectors is denoted as $(\underline{a} \wedge \underline{b})_i = \epsilon_{ijk} a_j b_k$.
The determinant of second-order tensors is labelled as $|\cdot|$; $|\boldsymbol{a}| = \epsilon_{ijk} \epsilon_{lmn} a_{il} a_{jm} a_{kn}$.
We also note: $(\nabla \underline{u})_{ij} = \partial_j u_i$, $\nabla \cdot \underline{u} = \partial_i u_i$, $(\nabla \wedge \underline{u})_i = \epsilon_{ijk} \partial_j u_k$,$(\nabla \cdot \boldsymbol{a})_j = \partial_i a_{ij}$.

3. The transformation procedure

3.1. The initial problem

A linear elastic body Ω, with the regular boundary $\partial\Omega$ and the moduli $\mathbb{C}(\underline{x})$, is subjected to volume forces $\underline{F}(\underline{x})$, to prescribed tractions $\underline{T}(\underline{x})$ on one part $\partial_T\Omega$ of the boundary and to prescribed displacements $\underline{U}(\underline{x})$ on its complement $\partial_U\Omega$ (Fig. 1a). The moduli C have the usual symmetry properties which read in an orthonormal basis $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$

$$\forall i, j, k, l; \quad C_{ijkl} = C_{ijlk} = C_{jikl} = C_{klij} \tag{1}$$

and $\mathbb{C}$ is positive definite, i.e.,

$$\forall \boldsymbol{\varepsilon} \text{ symmetric and } \boldsymbol{\varepsilon} \neq 0; \quad \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} > 0 \tag{2}$$

The resulting displacement field $\underline{u}(\underline{x})$ obeys the equilibrium equations and the boundary conditions, taking account of the constitutive equations:

$$\forall \underline{x} \in \Omega; \quad \nabla \cdot [\mathbb{C}(\underline{x}) : \nabla \underline{u}(\underline{x})] + \underline{F}(\underline{x}) = 0 \tag{3}$$

$$\forall \underline{x} \in \partial_T\Omega; \quad [\mathbb{C}(\underline{x}) : \nabla \underline{u}(\underline{x})] \cdot \underline{n}(\underline{x}) = \underline{T}(\underline{x}) \tag{4}$$

$$\forall \underline{x} \in \partial_U\Omega; \quad \underline{u}(\underline{x}) = \underline{U}(\underline{x}) \tag{5}$$

where $\underline{n}(\underline{x})$ is the unit outward normal to Ω at $\underline{x} \in \partial\Omega$.

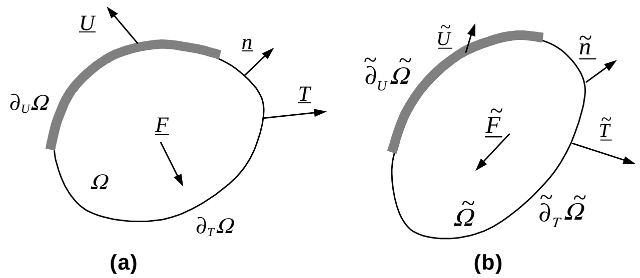

Fig. 1. The initial (a) and transformed (b) boundary value problems.

The associated variational formulation consists in finding $\underline{u}(\underline{x})$ obeying $\underline{u}(\underline{x}) = \underline{U}(\underline{x})$ on $\partial_U\Omega$ which minimizes the potential energy:

$$\mathsf{A}[\underline{u}] = \frac{1}{2}\int_\Omega \nabla\underline{u} : \mathbb{C} : \nabla\underline{u}\,\mathrm{d}\Omega - \int_\Omega \underline{F}\cdot\underline{u}\,\mathrm{d}\Omega - \int_{\partial_T\Omega} \underline{T}\cdot\underline{u}\,\mathrm{d}S \tag{6}$$

For sufficiently smooth functions, the variation of $\mathsf{A}[\underline{u}]$ associated to a small variation $\underline{w}$ of $\underline{u}$, namely $\underline{w}(\underline{x}) = \delta\underline{u}(\underline{x})$, which is given by

$$\delta\mathsf{A}[\underline{u},\underline{w}] = \int_\Omega \nabla\underline{u} : \mathbb{C} : \nabla\underline{w}\,\mathrm{d}\Omega - \int_\Omega \underline{F}\cdot\underline{w}\,\mathrm{d}\Omega - \int_{\partial_T\Omega} \underline{T}\cdot\underline{w}\,\mathrm{d}S \tag{7}$$

must then vanish

$$\forall\underline{w};\quad \{\forall\underline{x}\in\partial_U\Omega, \underline{w}(\underline{x}) = 0\} \Rightarrow \delta\mathsf{A}[\underline{u},\underline{w}] = 0 \tag{8}$$

3.2. The transformation

We consider a transformation defined on both $\underline{x}$ and $\underline{u}(\underline{x})$ independently:

$$\underline{\tilde{x}} = \underline{\varphi}(\underline{x}) \tag{9}$$

$$\underline{u}(\underline{x}) = \boldsymbol{Q}(\underline{x})\cdot\underline{\tilde{u}}(\underline{\tilde{x}}) \tag{10}$$

where we assume that $\boldsymbol{Q}(\underline{x})$ and $\underline{\varphi}(\underline{x})$ can be inverted. Let $\widetilde{\Omega}$ and $\tilde{\partial}\widetilde{\Omega}$ be the transformed domain and boundary for the space variable $\underline{\tilde{x}}$. We are looking for the conditions to be fulfilled by $\underline{\varphi}(\underline{x})$ and $\boldsymbol{Q}(\underline{x})$ for the transformed displacement field $\underline{\tilde{u}}(\underline{\tilde{x}})$ to be the solution of some boundary value problem for an elastic body $\widetilde{\Omega}$ with adequate moduli (Fig. 1b). Referring to (7), this will require the transformed functional of $\delta\mathsf{A}[\underline{u},\underline{w}]$ to have a similar expression, say $\delta\widetilde{\mathsf{A}}(\underline{\tilde{u}},\underline{\tilde{w}})$, with respect to the transformed fields $\underline{\tilde{u}}(\underline{\tilde{x}})$ and $\tilde{w}(\underline{\tilde{x}})$.

With evident notation and from classical results on the effect of a change of variables, the unit outward normal to $\tilde{\partial}\widetilde{\Omega}$, say $\underline{\tilde{n}}(\underline{\tilde{x}})$, is given by

$$\underline{\tilde{n}}(\underline{\tilde{x}}) = \kappa(\underline{x})\boldsymbol{R}^{-1}(\underline{x})\cdot\underline{n}(\underline{x}) \tag{11}$$

where the tensor $\boldsymbol{R}(\underline{x})$, whose inverse $\boldsymbol{R}^{-1}(\underline{x})$ is supposed to exist, and the scalar κ $(\underline{x})$ are defined by

$$\boldsymbol{R}(\underline{x}) = [\nabla\underline{\varphi}(\underline{x})]^{\mathrm{T}} \tag{12}$$

or more explicitly $R_{ij}(\underline{x}) = \partial_i\varphi_j(\underline{x})$, and

$$\underline{x}\in\partial_T\Omega;\quad \kappa(\underline{x}) = \|\boldsymbol{R}^{-1}(\underline{x})\cdot\underline{n}(\underline{x})\|^{-1} \tag{13}$$

The transformed line, surface and volume elements are related to the initial ones by $\mathrm{d}\underline{\tilde{x}} = \boldsymbol{R}^{\mathrm{T}}\cdot\mathrm{d}\underline{x}$, $\mathrm{d}\widetilde{\Omega} = J\,\mathrm{d}\Omega$ and $\mathrm{d}\widetilde{S} = J\kappa^{-1}\,\mathrm{d}S$, respectively, with

$$J(\underline{x}) = |\boldsymbol{R}(\underline{x})| \tag{14}$$

Since $\underline{w}(\underline{x})$ is arbitrary (with $\underline{w}(\underline{x}) = 0$, $\forall\underline{x}\in\partial_U\Omega$) in the variational Eq. (8), we can correlate $\underline{w}(\underline{x})$ and $\tilde{w}(\underline{\tilde{x}})$ through any invertible second-order tensor $\boldsymbol{M}(\underline{x})$ instead of $\boldsymbol{Q}(\underline{x})$. Referring to the last two terms of (7), we conclude that

$$\int_\Omega \underline{F}\cdot\underline{w}\,\mathrm{d}\Omega = \int_{\tilde{\Omega}} \underline{\widetilde{F}}\cdot\underline{\tilde{w}}\,\mathrm{d}\widetilde{\Omega} \tag{15}$$

$$\int_{\partial_T\Omega} \underline{T}\cdot\underline{w}\,\mathrm{d}S = \int_{\tilde{\partial}_T\tilde{\Omega}} \widetilde{T}\cdot\underline{\tilde{w}}\,\mathrm{d}\widetilde{S} \tag{16}$$

with

$$\underline{\widetilde{F}}(\tilde{x}) = J^{-1}\boldsymbol{M}^{\mathrm{T}}(\underline{x}) \cdot \underline{F}(\underline{x}) \tag{17}$$

$$\underline{\widetilde{T}}(\tilde{x}) = J^{-1}\kappa\boldsymbol{M}^{\mathrm{T}}(\underline{x}) \cdot \underline{T}(\underline{x}) \tag{18}$$

$$\underline{w}(\underline{x}) = \boldsymbol{M}(\underline{x}) \cdot \underline{\tilde{w}}(\tilde{x}) \tag{19}$$

In view of the coming analysis, the transformation of the first term of (7) can be performed more conveniently by using two sets of auxiliary displacement fields, say $\underline{\xi}^{(m)}(\underline{x})$ and $\underline{\xi}'^{(m)}(\underline{x})$, $m = 1, 2, 3$, defined by

$$\underline{\xi}^{(m)}(\underline{x}) = \boldsymbol{Q}(\underline{x}) \cdot \underline{e}_m \quad \underline{\xi}'^{(m)}(\underline{x}) = \boldsymbol{M}(\underline{x}) \cdot \underline{e}_m \tag{20}$$

where $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ are the three vectors of an orthonormal basis. Auxiliary stress fields $\boldsymbol{\Sigma}^{(m)}(\underline{x})$, $\boldsymbol{\Sigma}'^{(m)}(\underline{x})$ and body forces $\underline{F}^{(m)}(\underline{x})$, $\underline{F}'^{(m)}(\underline{x})$ can then be associated to these displacement fields through the definitions

$$\boldsymbol{\Sigma}^{(m)} = \mathbb{C} : \boldsymbol{\nabla}\underline{\xi}^{(m)}, \quad \underline{F}^{(m)} = -\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}^{(m)}, \quad \boldsymbol{\Sigma}'^{(m)} = \mathbb{C} : \boldsymbol{\nabla}\underline{\xi}'^{(m)}, \quad \underline{F}'^{(m)} = -\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}'^{(m)} \tag{21}$$

We also note

$$\underline{A}^{(mn)} = \boldsymbol{\Sigma}^{(m)} \cdot \underline{\xi}'^{(n)}, \quad \underline{A}'^{(mn)} = \boldsymbol{\Sigma}'^{(n)} \cdot \underline{\xi}^{(m)}, \quad \underline{\chi}^{(mn)} = \underline{A}^{(mn)} - \underline{A}'^{(mn)} \tag{22}$$

With the notation $\tilde{\partial}_i(\cdot) = \frac{\partial}{\partial \tilde{x}_i}(\cdot)$, we get: $\partial_i(\cdot) = R_{ij}\tilde{\partial}_j(\cdot)$. Consequently, we have the relations

$$\begin{aligned} \partial_i u_j &= (\partial_i Q_{jm})\tilde{u}_m + Q_{jm}R_{ip}\tilde{\partial}_p\tilde{u}_m \\ \partial_k w_l &= (\partial_k M_{\ln})\tilde{w}_n + M_{\ln}R_{kq}\tilde{\partial}_q\tilde{w}_n \end{aligned} \tag{23}$$

and then

$$\begin{aligned} &\boldsymbol{\nabla}\underline{u} : \mathbb{C} : \boldsymbol{\nabla}\underline{w} \\ &\quad = (\partial_i Q_{jm})C_{ijkl}(\partial_k M_{\ln})\tilde{u}_m\tilde{w}_n + Q_{jm}C_{ijkl}M_{\ln}\partial_i\tilde{u}_m\partial_k\tilde{w}_n + (\partial_i Q_{jm})C_{ijkl}M_{\ln}\tilde{u}_m\partial_k\tilde{w}_n \\ &\qquad + Q_{jm}C_{ijkl}(\partial_k M_{\ln})\partial_i\tilde{u}_m\tilde{w}_n \\ &\quad = \underline{F}^{(m)} \cdot \underline{\xi}'^{(n)}\tilde{u}_m\tilde{w}_n + Q_{jm}C_{ijkl}M_{\ln}\partial_i\tilde{u}_m\partial_k\tilde{w}_n + \boldsymbol{\nabla}.[\underline{A}^{(mn)}\tilde{u}_m\tilde{w}_n] - \chi_k^{(mn)}\partial_k\tilde{u}_m\tilde{w}_n \end{aligned} \tag{24}$$

The vector field $\underline{\chi}^{(mn)}$ can now be decomposed into its gradient and rotational parts, namely

$$\underline{\chi}^{(mn)} = \nabla\psi^{(mn)} + \nabla \wedge \underline{\Psi}^{(mn)} \quad (\text{i.e., } \chi_k^{(mn)} = \partial_k\psi^{(mn)} + \epsilon_{kij}\partial_i\Psi_j^{(mn)}) \tag{25}$$

With help of the relation $\epsilon_{kij}\partial_{ki}(\cdot) = 0$, the last term in (24) now reads

$$\chi_k^{(mn)}\partial_k\tilde{u}_m\tilde{w}_n = \partial_k\psi^{(mn)}\partial_k\tilde{u}_m\tilde{w}_n + \partial_i[\epsilon_{kij}\Psi_j^{(mn)}\partial_k\tilde{u}_m\tilde{w}_n] - \epsilon_{kij}\Psi_j^{(mn)}\partial_k\tilde{u}_m\partial_i\tilde{w}_n$$

so that (24) becomes

$$\boldsymbol{\nabla}\underline{u} : \mathbb{C} : \boldsymbol{\nabla}\underline{w} = [F^{(m)} \cdot \underline{\xi}'^{(n)}\tilde{u}_m - \nabla\psi^{(mn)}.\boldsymbol{\nabla}\tilde{u}_m]\tilde{w}_n + D_{imkn}\partial_i\tilde{u}_m\partial_k\tilde{w}_n + \boldsymbol{\nabla}.\{[\underline{A}^{(mn)}\tilde{u}_m - \underline{\Psi}^{(mn)} \wedge \boldsymbol{\nabla}\tilde{u}_m]\tilde{w}_n\} \tag{26}$$

with

$$D_{imkn} = Q_{jm}C_{ijkl}M_{\ln} - \epsilon_{kij}\Psi_j^{(mn)} \tag{27}$$

Integration of (26) then leads, by taking account of $\underline{w} = 0$ on $\partial_U\Omega$ and after some reduction, to

$$\begin{aligned} &\int_\Omega \boldsymbol{\nabla}\underline{u} : \mathbb{C} : \boldsymbol{\nabla}\underline{w}\,\mathrm{d}\Omega \\ &\qquad = \int_\Omega [\underline{F}^{(m)} \cdot \underline{\xi}'^{(n)}\tilde{u}_m - \nabla\psi^{(mn)} \cdot \nabla\tilde{u}_m]\tilde{w}_n\,\mathrm{d}\Omega + \int_{\partial_T\Omega} [\underline{A}^{(mn)}\tilde{u}_m - \underline{\Psi}^{(mn)} \wedge \boldsymbol{\nabla}\tilde{u}_m] \cdot \underline{n}\tilde{w}_n\,\mathrm{d}S \\ &\qquad\quad + \int_{\tilde{\Omega}} \widetilde{C}_{mpnq}\tilde{\partial}_p\tilde{u}_m\tilde{\partial}_q\tilde{w}_n\,\mathrm{d}\widetilde{\Omega} \end{aligned} \tag{28}$$

with

$$\widetilde{C}_{mpnq} = J^{-1} D_{imkn} R_{ip} R_{kq} \tag{29}$$

where we can also write

$$\int_{\tilde{\Omega}} \widetilde{C}_{mpnq} \tilde{\partial}_p \tilde{u}_m \tilde{\partial}_q \tilde{w}_n \, \mathrm{d}\widetilde{\Omega} = \int_{\Omega} \tilde{\boldsymbol{\nabla}} \underline{\tilde{u}} : \widetilde{\mathbb{C}} : \tilde{\boldsymbol{\nabla}} \underline{\tilde{w}} \, \mathrm{d}\widetilde{\Omega} \tag{30}$$

Finally, by transferring (15), (16), (28) and (30) into (7), we find

$$\begin{aligned} \delta \mathsf{A}[\underline{u}, \underline{w}] = & \int_{\Omega} [\underline{F}^{(m)} \cdot \underline{\xi}'^{(n)} \tilde{u}_m - \nabla \psi^{(mn)} . \boldsymbol{\nabla} \tilde{u}_m] \tilde{w}_n \, \mathrm{d}\Omega + \int_{\partial_T \Omega} [\underline{A}^{(mn)} \tilde{u}_m - \underline{\Psi}^{(mn)} \wedge \boldsymbol{\nabla} \tilde{u}_m] \cdot \underline{n} \tilde{w}_n \, \mathrm{d}S \\ & + \int_{\tilde{\Omega}} \tilde{\boldsymbol{\nabla}} \underline{\tilde{u}} : \widetilde{\mathbb{C}} : \tilde{\boldsymbol{\nabla}} \underline{\tilde{w}} \mathrm{d}\widetilde{\Omega} - \int_{\tilde{\Omega}} \underline{\widetilde{F}} \cdot \underline{\tilde{w}} \mathrm{d}\widetilde{\Omega} - \int_{\tilde{\partial}_T \tilde{\Omega}} \underline{\widetilde{T}} \cdot \underline{\tilde{w}} \mathrm{d}\widetilde{S} \end{aligned} \tag{31}$$

Note that the change of coordinates has not been performed completely in (31) ($\underline{u}$ and $\underline{w}$ only have been transformed and the two first integrals still depend on $\underline{x}$): this will make some of the following derivations easier.

3.3. The transformed problem

In order to ensure the nature of the transformed problem to be the same as that of the original one, i.e., an elastic boundary value problem, we have now to assign $\delta\mathsf{A}(\underline{u}, \underline{w})$, as given by (31), to be, like in (7), the sum of a symmetric bilinear form for $\tilde{\boldsymbol{\nabla}}\underline{\tilde{u}}$ and $\tilde{\boldsymbol{\nabla}}\underline{\tilde{w}}$, with $\widetilde{\mathbb{C}}$ obeying (1) and (2), integrated over $\widetilde{\Omega}$, and of two linear forms for $\underline{\tilde{w}}$, with coefficients independent of $\underline{\tilde{u}}$, integrated on $\widetilde{\Omega}$ and $\tilde{\partial}_T\widetilde{\Omega}$, namely

$$\delta \widetilde{\mathsf{A}}(\underline{\tilde{u}}, \underline{\tilde{w}}) = \int_{\Omega} \tilde{\boldsymbol{\nabla}} \underline{\tilde{u}} : \widetilde{\mathbb{C}} : \tilde{\boldsymbol{\nabla}} \underline{\tilde{w}} \mathrm{d}\widetilde{\Omega} - \int_{\tilde{\Omega}} \underline{\widetilde{F}} \cdot \underline{\tilde{w}} \mathrm{d}\widetilde{\Omega} - \int_{\tilde{\partial}_T \tilde{\Omega}} \underline{\widetilde{T}} \cdot \underline{\tilde{w}} \mathrm{d}\widetilde{S} \tag{32}$$

The condition $\underline{u}(\underline{x}) = \underline{U}(\underline{x})$ on $\partial_U\Omega$ has been transformed into $\underline{\tilde{u}}(\underline{\tilde{x}}) = \underline{\widetilde{U}}(\underline{\tilde{x}})$ on $\tilde{\partial}_U\widetilde{\Omega}$ with $\underline{U}(\underline{x}) = \boldsymbol{Q}(\underline{x}) \cdot \underline{\widetilde{U}}(\underline{\tilde{x}})$. Since $\boldsymbol{M}(\underline{x})$ is invertible, $\underline{\tilde{w}}(\underline{\tilde{x}})$ vanishes on $\underline{\tilde{x}} \in \tilde{\partial}_U\widetilde{\Omega}$ as soon as $\underline{w}(\underline{x})$ does so on $\partial_U\Omega$. The variational Eq. (8) is then equivalent to

$$\forall \underline{\tilde{w}}; \quad \{\forall \underline{\tilde{x}} \in \tilde{\partial}_U \widetilde{\Omega}, \underline{\tilde{w}}(\underline{\tilde{x}}) = 0\} \Rightarrow \delta \widetilde{\mathsf{A}}[\underline{\tilde{u}}, \underline{\tilde{w}}] = 0 \tag{33}$$

So, in order to solve the initial problem, it will be equivalent to do so for the transformed one by deriving the solution $\underline{\tilde{u}}(\underline{\tilde{x}})$ from the variational Eq. (33), with $\underline{\tilde{u}}(\underline{\tilde{x}}) = \underline{\widetilde{U}}(\underline{\tilde{x}})$ on $\tilde{\partial}_U\widetilde{\Omega}$: from that, the solution $\underline{u}(\underline{x})$ of the initial problem will be derived through the inverse transformation. As it is well known, this method is equivalent to the direct resolution of the field equations

$$\begin{cases} \forall \underline{\tilde{x}} \in \widetilde{\Omega}; & \tilde{\boldsymbol{\nabla}} . [\widetilde{\mathbb{C}}(\underline{\tilde{x}}) : \tilde{\boldsymbol{\nabla}} \underline{\tilde{u}}(\underline{x})] + \underline{\widetilde{F}}(\underline{\tilde{x}}) = 0 \\ \forall \underline{\tilde{x}} \in \tilde{\partial}_T \widetilde{\Omega}; & [\widetilde{\mathbb{C}}(\underline{\tilde{x}}) : \tilde{\boldsymbol{\nabla}} \underline{\tilde{u}}(\underline{x})] \cdot \underline{\tilde{n}}(\underline{\tilde{x}}) = \underline{\widetilde{T}}(\underline{\tilde{x}}) \end{cases}$$
$$\forall \underline{\tilde{x}} \in \tilde{\partial}_U \widetilde{\Omega}; \quad \underline{\tilde{u}}(\underline{\tilde{x}}) = \underline{\widetilde{U}}(\underline{\tilde{x}}) \tag{34}$$

Let us now focus on the derivation of sufficient conditions for the considered transformation to be admissible. This can be done by making the two first integrals in (31) vanish for any $\underline{u}$, with $\widetilde{\mathbb{C}}$, defined by (29), obeying (1) and (2). These conditions first imply

$$\forall m, n, \ \forall \underline{x} \in \Omega; \quad \underline{F}^{(m)} \cdot \underline{\xi}'^{(n)} = 0, \quad \nabla \psi^{(mn)} = 0 \tag{35}$$

$$\forall m, n, \ \forall \underline{x} \in \partial_T \Omega; \quad \underline{A}^{(mn)} \cdot \underline{n} = 0, \quad \underline{n} \wedge \underline{\Psi}^{(mn)} = 0 \tag{36}$$

Since $\boldsymbol{M}^{-1}$ exists, $\underline{\xi}'^{(n)}$ are three independent vectors; so, according to the first condition in (35), $\underline{F}^{(m)} = 0$. Similarly, the first condition in (36) implies $\underline{T}^{(m)} = \boldsymbol{\Sigma}^{(m)} \cdot \underline{n} = 0$ on $\partial_T \Omega$. The second condition in (35) leads to

$$\underline{\chi}^{(mn)} = \nabla \wedge \underline{\Psi}^{(mn)} \tag{37}$$

which implies that $\nabla \cdot \underline{\chi}^{(mn)} = 0$. Since we can write

$$\nabla \cdot \underline{\chi}^{(mn)} = -\underline{F}^{(m)} \cdot \underline{\xi}'^{(n)} + \underline{F}'^{(n)} \cdot \underline{\xi}^{(m)} + \boldsymbol{\Sigma}^{(m)} : \boldsymbol{\nabla}\underline{\xi}'^{(n)} - \boldsymbol{\Sigma}'^{(n)} : \boldsymbol{\nabla}\underline{\xi}^{(m)} = \underline{F}'^{(n)} \cdot \underline{\xi}^{(m)}$$

due to the symmetry of $\mathbb{C}$ and to the condition $\underline{F}^{(m)} = 0$, we must have $\underline{F}'^{(n)} \cdot \underline{\xi}^{(m)} = 0$, which implies $\underline{F}'^{(n)} = 0$ (since $\boldsymbol{Q}$ is invertible). Conditions (35) and (36) can then be replaced by

$$\forall m,\ \forall \underline{x} \in \Omega; \quad \underline{F}^{(m)} = 0, \quad \underline{F}'^{(m)} = 0 \tag{38}$$

$$\forall m, n,\ \forall \underline{x} \in \partial_T \Omega; \quad \boldsymbol{\Sigma}^{(m)} \cdot \underline{n} = 0, \quad \underline{n} \wedge \underline{\Psi}^{(mn)} = 0 \tag{39}$$

In addition, $\widetilde{\mathbb{C}}$ must obey (1) and (2). As for the symmetry conditions, namely

$$\forall m, n, p, q,\ \forall \underline{x} \in \Omega; \quad \widetilde{C}_{mpnq} = \widetilde{C}_{mpqn} = \widetilde{C}_{pmnq} = \widetilde{C}_{nqmp} \tag{40}$$

we can first notice that, according to the definition (29), the diagonal symmetry condition, i.e., $\widetilde{C}_{mpnq} = \widetilde{C}_{nqmp}$, needs

$$\forall i, k, m, n,\ \forall \underline{x} \in \Omega; \quad D_{imkn} = D_{knim} \tag{41}$$

or equivalently

$$\forall i, k, m, n,\ \forall \underline{x} \in \Omega; \quad C_{ijkl}[Q_{jm} M_{\ln} - Q_{\ln} M_{jm}] = \epsilon_{kij}[\Psi_j^{(nm)} + \Psi_j^{(mn)}] \tag{42}$$

If this relation is obeyed, the property $\widetilde{C}_{mpnq} = \widetilde{C}_{pmnq}$ would result from the condition $\widetilde{C}_{mpnq} = \widetilde{C}_{mpqn}$. From (29), this condition reduces to

$$\forall i, q, m, n,\ \forall \underline{x} \in \Omega; \quad D_{imkn} R_{kq} = D_{imkq} R_{kn} \tag{43}$$

Finally, according to (2), $\widetilde{\mathbb{C}}$ must be positive definite.

So, a whole set of sufficient conditions for the proposed transformation, defined by $\boldsymbol{Q}(\underline{x})$ and $\underline{\varphi}(\underline{x})$ according to (9) and (10), to be admissible is the following:

- to choose the invertible tensor field $\boldsymbol{Q}(\underline{x})$ in such a way that the three displacement fields $\underline{\xi}^{(m)}(\underline{x}) = \boldsymbol{Q}(\underline{x}) \cdot \underline{e}_m$ can satisfy the equations

$$\forall \underline{x} \in \Omega;\ \nabla \cdot [\mathbb{C} : \boldsymbol{\nabla}\underline{\xi}^{(m)}] = 0, \quad \forall \underline{x} \in \partial_T \Omega;\ [\mathbb{C} : \boldsymbol{\nabla}\underline{\xi}^{(m)}] \cdot \underline{n} = 0 \tag{44}$$

- to find an invertible tensor field $\boldsymbol{M}(\underline{x})$ such that the associated displacement fields $\underline{\xi}'^{(m)}(\underline{x}) = \boldsymbol{M}(\underline{x}) \cdot \underline{e}_m$ obey

$$\forall \underline{x} \in \Omega; \quad \nabla \cdot [\mathbb{C} : \boldsymbol{\nabla}\underline{\xi}'^{(m)}] = 0 \tag{45}$$

- this ensures that $\underline{\chi}^{(mn)} = [\mathbb{C} : \boldsymbol{\nabla}\underline{\xi}^{(m)}] \cdot \underline{\xi}'^{(n)} - [\mathbb{C} : \boldsymbol{\nabla}\underline{\xi}'^{(n)}] \cdot \underline{\xi}^{(m)}$ is a rotational field, i.e., that there exists vector fields $\underline{\Psi}^{(mn)}$ such that $\underline{\chi}^{(mn)} = \nabla \wedge \underline{\Psi}^{(mn)}$; we have then to choose one such vector field $\underline{\Psi}^{(mn)}$, obeying

$$\forall \underline{x} \in \partial_T \Omega; \quad \underline{n} \wedge \underline{\Psi}^{(mn)} = 0 \tag{46}$$

- with $\mathbb{D}$ defined by (27), to choose the vector field $\underline{\varphi}(\underline{x})$ so that, with $R_{ij} = \partial_i \varphi_j$, (42) and (43) are verified and $\widetilde{\mathbb{C}}$, defined by (29), is positive definite.

Note that if $\boldsymbol{M}(\underline{x})$ is chosen as $\boldsymbol{M}(\underline{x}) = q\boldsymbol{Q}(\underline{x})$ with q a constant, (45) is satisfied as soon as $\boldsymbol{Q}(\underline{x})$ obeys the first condition (44); we also have $\underline{\chi}^{(mn)} = -\underline{\chi}^{(nm)}$ and if $\underline{\Psi}^{(mn)}$ is chosen antisymmetric too, (42) is satisfied automatically. The condition to be satisfied by $\underline{\varphi}$ and $\boldsymbol{Q}$ would in this case reduce to (44), (46), (44) and definite positivity of $\widetilde{\mathbb{C}}$. Conversely, it can be shown that if the tensor $\overline{\mathbb{C}}$ defined by

$$\overline{C}_{ikjl} = (C_{ijkl} + C_{kjil})/2 \tag{47}$$

is definite, i.e., satisfies

$$\overline{\mathbb{C}} : \boldsymbol{a} = 0 \Rightarrow \boldsymbol{a} = 0 \tag{48}$$

for any symmetric second-order tensor $\boldsymbol{a}$, which occurs especially for isotropic moduli $\boldsymbol{C}$, then $\boldsymbol{M}(\underline{x})$ necessarily must have the form $\boldsymbol{M}(\underline{x}) = q\boldsymbol{Q}(\underline{x})$ with q a constant (see Appendix A).

Finally, we cannot certify at the moment that the above conditions, though they are the softest we have been able to find, are the most general ones (i.e., both sufficient and necessary) for the transformation defined by (9) and (10) to transform an elastic boundary value problem into another one. Further investigations on this point, as well as on extended applications of the proposed method to arbitrary anisotropic heterogeneous bodies are still in progress. In what follows, we focus on the particular case of homogeneous transformations and Saint-Venant-type anisotropy.

4. Homogeneous transformations

4.1. A simple transformation

A simple example of transformation obeying the conditions derived hereabove can be given by choosing $\boldsymbol{Q}$ as a constant (invertible) tensor. This corresponds to constant $\underline{\xi}^{(m)}$ vectors and to $\boldsymbol{\Sigma}^{(m)} = 0$; $\boldsymbol{M} = q\boldsymbol{Q}$ is constant too with $q = |\boldsymbol{Q}|$. Thus, $\underline{\Psi}^{(mn)}$ can be taken as zero and $\underline{\varphi}(\underline{x}) = \boldsymbol{Q}^{\mathrm{T}} \cdot \underline{x}$. So, $\boldsymbol{R} = \boldsymbol{Q}$. With $\boldsymbol{P} = (\boldsymbol{R}^{\mathrm{T}})^{-1}$, relations (9) and (10) now read

$$\begin{cases} \underline{x} = \boldsymbol{P} \cdot \underline{\tilde{x}} \\ \underline{u}(\underline{x}) = \boldsymbol{Q} \cdot \underline{\tilde{u}}(\underline{\tilde{x}}) \\ \boldsymbol{Q} = (\boldsymbol{P}^{\mathrm{T}})^{-1} \end{cases} \tag{49}$$

The transformation Eqs. (29), (17) and (18) reduce to

$$\widetilde{C}_{mnpq} = C_{ijkl} Q_{im} Q_{jn} Q_{kp} Q_{lq} \tag{50}$$

$$\underline{\widetilde{F}}(\underline{\tilde{x}}) = \boldsymbol{Q}^{\mathrm{T}} \cdot \underline{F}(\underline{x}), \quad \underline{\widetilde{T}}(\underline{\tilde{x}}) = \kappa \boldsymbol{Q}^{\mathrm{T}} \cdot \underline{T}(\underline{x}), \quad \kappa = \|\boldsymbol{P}^{\mathrm{T}} \cdot \underline{n}\|^{-1} \tag{51}$$

The strain and stress fields of the original and transformed problems are linked by the relations

$$\tilde{\boldsymbol{\varepsilon}}(\underline{\tilde{x}}) = \boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{\varepsilon}(x) \cdot \boldsymbol{P}, \quad \tilde{\boldsymbol{\sigma}}(\underline{\tilde{x}}) = \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{\sigma}(x) \cdot \boldsymbol{Q} \tag{52}$$

and the transformation proposed initially by Pouya (2000) is recovered.

Note that Eq. (50) has been used by Olver (1988) in order to reduce the number of canonical elastic moduli; he also showed that this equation cannot transform an arbitrary type of anisotropy into isotropy. As mentioned in the Introduction, special applications of this transformation in the case of zero volume forces and boundary tractions, so defined by (49), (50) and (52) have been developed by Milgrom and Shtrikman (1992) or by Milton (2002).

Let us stress the fact that, in general, (49) does not reduce to a change of coordinates since $\underline{x}$ and $\underline{u}$ do not transform in the same way. If $\boldsymbol{P}$ is orthogonal (i.e., $\boldsymbol{P} = (\boldsymbol{P}^{\mathrm{T}})^{-1}$), then $\boldsymbol{Q} = \boldsymbol{P}$: the transformation is degenerated into a change of orthonormal basis and the constitutive behavior is unchanged. In the general case, $\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{T} \cdot \boldsymbol{S}$, where $\boldsymbol{T}$ is orthogonal and $\boldsymbol{S}$ is symmetric. The total transformation can be decomposed into two operations: the first one, defined by $\boldsymbol{T}$, transforms $(\underline{x}, \underline{u})$ into $(\underline{x}', \underline{u}')$ and the second one, defined by $\boldsymbol{S}$, transforms $(\underline{x}', \underline{u}')$ into $(\tilde{\underline{x}}, \tilde{\underline{u}})$:

$$x = \boldsymbol{T} \cdot \underline{x}' = \boldsymbol{T} \cdot \boldsymbol{S} \cdot \tilde{\underline{x}}$$
$$\underline{u} = \boldsymbol{T} \cdot \underline{u}' = \boldsymbol{T} \cdot \boldsymbol{S}^{-1} \cdot \tilde{\underline{u}}$$

The transformation $\boldsymbol{T}$ does not lead to any new results; so we can restrict ourselves to transformations defined by symmetric, positive definite matrices $\boldsymbol{S}$. Two main preliminary questions must be addressed in view of definite applications: the transformation of the geometry and boundary conditions on the one hand and the nature of the anisotropy of the elastic moduli on the other hand.

4.2. Preliminary analysis

4.2.1. Geometry and boundary conditions

In the following, we consider only problems with a geometry which is invariant or belongs to a family which is invariant under a linear transformation of the coordinates: the Green tensors for an infinite or a semi-infinite body, the ellipsoidal inclusion in an infinite matrix, etc.

4.2.1.1. The inclusion problem. The transformation (49) has been applied to the inclusion problem by Milgrom and Shtrikman (1992) and, independently, to the inhomogeneous inclusion problem by Pouya (2000). For this case, a bounded elastic inhomogeneity Ω, with the regular boundary $\partial\Omega$, the elastic moduli $\mathbb{C}^{(1)}$ and the (possibly nonuniform) eigenstrain $\boldsymbol{\varepsilon}^0$ is perfectly embedded in an infinite matrix $\mathscr{M}$ with the moduli $\mathbb{C}^{(2)}$. The eigenstrain $\boldsymbol{\varepsilon}^0$ is supposed to derive from the displacement field $\underline{u}^0$, defined in Ω, and the matrix $\mathscr{M}$ is subjected to the uniform strain $\boldsymbol{E}^\infty$ at infinity. The unknown displacement fields $\underline{u}^{(1)}$ in Ω and $\underline{u}^{(2)}$ in $\mathscr{M}$ obey the following conditions:

$$\begin{cases} \forall \underline{x} \in \Omega; & C^{(1)}_{ijkl}\partial_{jk}[u^{(1)}_l(\underline{x}) - u^0_l(\underline{x})] = 0 \\ \forall \underline{x} \in \mathscr{M}; & C^{(2)}_{ijkl}\partial_{jk}u^{(2)}_l(\underline{x}) = 0 \\ \forall \underline{x} \in \partial\Omega; & \underline{u}^{(1)}(\underline{x}) = \underline{u}^{(2)}(\underline{x}) \\ \forall \underline{x} \in \partial\Omega; & n_j(\underline{x})C^{(1)}_{ijkl}\partial_k[u^{(1)}_l(\underline{x}) - u^0_l(\underline{x})] = n_j(\underline{x})C^{(2)}_{ijkl}\partial_k u^{(2)}_l(\underline{x}) \\ & \lim\limits_{\|\underline{x}\|\to\infty} [u^{(2)}_i(\underline{x}) - E^\infty_{ij}x_j] = 0 \end{cases} \tag{53}$$

This problem can be transformed according to (49) through the invertible tensor $\boldsymbol{P}$ operating on $\underline{x}$, $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$. The transformed displacement fields $\tilde{\underline{u}}^{(1)}(\tilde{\underline{x}})$ and $\tilde{\underline{u}}^{(2)}(\tilde{\underline{x}})$ obey the same (53) with the transformed moduli $\widetilde{\mathbb{C}}^{(1)}$ and $\widetilde{\mathbb{C}}^{(2)}$ derived from $\mathbb{C}^{(1)}$ and $\mathbb{C}^{(2)}$ by relations similar to (50) and with the conditions $\tilde{\underline{u}}^0 = \boldsymbol{P}^{\mathrm{T}} \cdot \underline{u}^0$ or $\tilde{\varepsilon}^0 = \boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{\varepsilon}^0 \cdot \boldsymbol{P}$ and $\widetilde{\boldsymbol{E}}^\infty = \boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{E}^\infty \cdot \boldsymbol{P}$. When Ω is an ellipsoid and $\boldsymbol{\varepsilon}^0$ is uniform in it, it is well known that $\boldsymbol{\varepsilon}^{(1)}$, the strain field in the inclusion, is uniform too and is a linear function of $\boldsymbol{\varepsilon}^0$. If, in addition, $\mathbb{C}^{(1)} = \mathbb{C}^{(2)}$ and $\boldsymbol{E}^\infty = 0$, the solution reads

$$\boldsymbol{\varepsilon}^{(1)} = \mathbb{S}^E : \boldsymbol{\varepsilon}^0 \tag{54}$$

where $\mathbb{S}^E$ is the "Eshelby tensor" (1957). The solution of the transformed problem has the same properties and, according to (50), the transformed Eshelby tensor $\widetilde{\mathbb{S}}^E$ is connected with $\mathbb{S}^E$ by (Pouya, 2000)

$$\widetilde{S}^E_{ijkl} = S^E_{mnpq}P_{mi}P_{nj}Q_{pk}Q_{ql} \tag{55}$$

Note that $\mathbb{S}^E$ and $\widetilde{\mathbb{S}}^E$ not only correspond to different moduli but also to different inclusion shapes: if the ellipsoid surface $\partial\Omega$ is defined by $\underline{x} \cdot \boldsymbol{H} \cdot \underline{x} = 1$, with $\boldsymbol{H}$ a symmetric, positive definite second-order tensor, the boundary of the transformed inclusion is given by $\underline{\tilde{x}} \cdot \boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{H} \cdot \boldsymbol{P} \cdot \underline{\tilde{x}} = 1$. The Eshelby tensor has a known closed form expression for isotropy (Eshelby, 1957) and transverse isotropy (Kneer, 1965; Mura, 1982). Thanks to (55), this can be extended, as illustrated below, to more general symmetries for which $\widetilde{\mathbb{C}}^{(2)}$, instead of $\mathbb{C}^{(2)}$, is isotropic or transversely isotropic.

This transformation can also be used in order to transform the ellipsoidal shape of the inclusion into a spherical one: $\boldsymbol{P}$ has to be such that $\boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{H} \cdot \boldsymbol{P} = R^{-2}\boldsymbol{I}$, with $\boldsymbol{I}$ the second-order unity tensor and R the sphere radius. This may be useful in cases when $\mathbb{C}$ is arbitrary: for instance, the numerical method based on the Fourier transform (Mura, 1982) can be simplified in this way and it is no more necessary to deal with spherical and ellipsoidal inclusions separately, as in Mura (1982).

4.2.1.2. Green functions. A similar treatment can be applied to the derivation of the Green tensor $\boldsymbol{G}(\underline{x}, \underline{x}')$ which, for an elastic body, correlates the displacement field $\underline{u}(\underline{x})$ generated by a point force $\underline{F}(\underline{x}) = \underline{\phi}\delta(\underline{x} - \underline{x}')$ applied at $\underline{x}'$ to this force:

$$\underline{u}(\underline{x}) = \boldsymbol{G}(\underline{x}, \underline{x}') \cdot \underline{\phi} \tag{56}$$

The $\boldsymbol{P}$-transformed problem is that of an elastic body with a modified geometry and the moduli $\widetilde{\mathbb{C}}$ given by (50), subjected, according to (51), to the force density

$$\underline{\widetilde{F}}(\underline{\tilde{x}}) = \boldsymbol{P}^{-1} \cdot \underline{F}(\underline{x}) = \boldsymbol{P}^{-1} \cdot \underline{\phi}\delta(\underline{x} - \underline{x}') = |\boldsymbol{P}^{-1}|\boldsymbol{P}^{-1} \cdot \underline{\phi}\delta(\underline{\tilde{x}} - \underline{\tilde{x}}') = \underline{\tilde{\phi}}\delta(\underline{\tilde{x}} - \underline{\tilde{x}}') \tag{57}$$

with $\underline{\tilde{\phi}} = |\boldsymbol{P}^{-1}|\boldsymbol{P}^{-1} \cdot \underline{\phi}$. The resulting displacement field $\underline{\tilde{u}}(\underline{\tilde{x}}) = \boldsymbol{P}^{\mathrm{T}} \cdot \underline{u}(\underline{x})$ leads, through the definition (56) applied to the transformed Green tensor $\widetilde{G}(\underline{\tilde{x}}, \underline{\tilde{x}}')$, to the relation

$$\widetilde{\boldsymbol{G}}(\underline{\tilde{x}}, \underline{\tilde{x}}') = |\boldsymbol{P}|\boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{G}(\underline{x}, \underline{x}') \cdot \boldsymbol{P} \tag{58}$$

For an infinite body, the geometry is not modified by the transformation: if $\widetilde{\boldsymbol{G}}(\underline{\tilde{x}}, \underline{\tilde{x}}')$ can be calculated in closed form, (58) gives access to $\boldsymbol{G}(\underline{x}, \underline{x}')$. Note that in that case both $\boldsymbol{G}$ and $\widetilde{\boldsymbol{G}}$ depend on $(\underline{\tilde{x}} - \underline{\tilde{x}}')$ only. For a semi-infinite body defined by $\underline{x} \cdot \underline{n} \geqslant 0$ with $\underline{n}$ the outward unit normal to the plane boundary, the geometrical transformation also involves a rotation: the transformed body is the half space defined by $\underline{\tilde{x}} \cdot \underline{\widetilde{N}} \geqslant 0$ with $\underline{\widetilde{N}} = \boldsymbol{P}^{\mathrm{T}} \cdot \underline{n}$. Here again, (58) makes the derivation of $\boldsymbol{G}$ possible as soon as $\widetilde{\boldsymbol{G}}$ is known.

4.2.2. Anisotropy

The proposed transformation can then be used in order to extend known solutions for isotropic or transverse isotropic elasticity toward more general situations.

4.2.2.1. Transformed isotropy. The question to be answered is: which must be the symmetry of the initial moduli so that the transformed ones are isotropic? According to (50), which can be inverted into

$$C_{ijkl} = P_{im}P_{jn}P_{kp}P_{lq}\widetilde{C}_{mnpq} \tag{59}$$

and with isotropic moduli $\widetilde{\mathbb{C}}$ expressed as a function of Lamé coefficients λ and μ

$$\widetilde{C}_{mnpq} = \lambda\delta_{mn}\delta_{pq} + \mu(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}) \tag{60}$$

the initial moduli $\mathbb{C}$ must read (Milgrom and Shtrikman, 1992; Pouya, 2000[1]):

$$C_{ijkl} = \lambda D_{ij}D_{kl} + \mu(D_{ik}D_{jl} + D_{il}D_{jk}), \quad \boldsymbol{D} = \boldsymbol{P} \cdot \boldsymbol{P}^{\mathrm{T}} \tag{61}$$

[1] Note that in Pouya (2000) a misprint has unfortunately changed D_{ij} into P_{ij}.

The anisotropy of these moduli is a special case of orthotropy: an eigenbasis of $\boldsymbol{D}$ is an orthotropy basis for $\mathbb{C}$. In such a basis, with d_α ($\alpha = 1,2,3$) the (positive) eigenvalues of $\boldsymbol{D}$, $a_\alpha = \sqrt{d_\alpha}$, $c_{\alpha\alpha} = (\lambda + 2\mu)(a_\alpha)^4$ (without summation on α) and $\eta = \lambda/(\lambda + 2\mu)$, the 6×6 matrix of the elastic moduli for $\mathbb{C}$, in the Voight's notation, reads, according to (61)

$$[\boldsymbol{C}]^{\mathrm{TrI}} = \begin{bmatrix} c_{11} & \eta\sqrt{c_{11}c_{22}} & \eta\sqrt{c_{11}c_{33}} & & & \\ & c_{22} & \eta\sqrt{c_{22}c_{33}} & & & \\ & & c_{33} & & & \\ & & & \frac{1-\eta}{2}\sqrt{c_{22}c_{33}} & & \\ & & & & \frac{1-\eta}{2}\sqrt{c_{11}c_{33}} & \\ & & & & & \frac{1-\eta}{2}\sqrt{c_{11}c_{22}} \end{bmatrix} \tag{62}$$

With the notation

$$(\alpha = 1,2,3) \quad E_\alpha = \frac{3\lambda + 2\mu}{\lambda + \mu}\mu(a_\alpha)^4, \quad v = \lambda/[2(\lambda + \mu)] \tag{63}$$

the compliance matrix can be written as

$$[\mathbf{S}]^{\mathrm{TrI}} = \begin{bmatrix} \frac{1}{E_1} & \frac{-v}{\sqrt{E_1E_2}} & \frac{-v}{\sqrt{E_1E_3}} & & & \\ \frac{-v}{\sqrt{E_1E_2}} & \frac{1}{E_2} & \frac{-v}{\sqrt{E_2E_3}} & & & \\ \frac{-v}{\sqrt{E_1E_3}} & \frac{-v}{\sqrt{E_2E_3}} & \frac{1}{E_3} & & & \\ & & & \frac{2(1+v)}{\sqrt{E_2E_3}} & & \\ & & & & \frac{2(1+v)}{\sqrt{E_3E_1}} & \\ & & & & & \frac{2(1+v)}{\sqrt{E_1E_2}} \end{bmatrix} \tag{64}$$

This special case of anisotropy depends on four independent parameters only, namely (c_{11}, c_{22}, c_{33}, η) or (E_1, E_2, E_3, v). It is referred as TrI (transformed isotropy) in the diagram of Fig. 3. Conversely, if the moduli or compliances of a given material can be written in the form (62) or (64), the tensor $\mathbb{C}$ must have the form (61): as a matter of fact, from (c_{11}, c_{22}, c_{33}, η), one can define ($\alpha = 1,2,3$, no summation on α)

$$\mu = c(1-\eta)/2, \quad \lambda = 2\eta\mu/(1-\eta), \quad a_\alpha = (c_{\alpha\alpha}/c)^{1/4} \tag{65}$$

where c is an arbitrary positive constant, or, from (E_1, E_2, E_3, v),

$$\mu = E/[2(1+v)], \quad \lambda = Ev/[(1+v)(1-2v)], \quad a_\alpha = (E_\alpha/E)^{1/4} \tag{66}$$

where E is an arbitrary positive constant. If $\boldsymbol{P}$ is, in the orthotropy basis, the diagonal tensor with eigenvalues a_α, and $\boldsymbol{D} = \boldsymbol{P} \cdot \boldsymbol{P}^{\mathrm{T}}$, the tensor $\mathbb{C}$ reads as in (61) and then it is transformed by $\boldsymbol{P}$ into an isotropic tensor.

Note that this special case of orthotropy, as defined by (62) or (64), has already been considered by de Saint Venant (1863). Let the modulus and compliance along the direction $\underline{n}$ be defined by

$$c(\underline{n}) = (\underline{n} \otimes \underline{n}) : \mathbb{C} : (\underline{n} \otimes \underline{n}), \quad 1/E(\underline{n}) = (\underline{n} \otimes \underline{n}) : \mathbb{S} : (\underline{n} \otimes \underline{n})$$

respectively; de Saint-Venant was concerned with elastic materials for which, in spherical coordinates $(r, \underline{n})$, either the surface $r(\underline{n}) = \sqrt[4]{E(\underline{n})}$ or $r(\underline{n}) = (\sqrt[4]{c(\underline{n})})^{-1}$ has an ellipsoidal shape (these classes of anisotropy are denoted SV1 and SV2, respectively, in the diagram of Fig. 3). He found that this is the case for both these surfaces for materials defined by (62) or (64), so that we can say that TrI = SV1 ∩ SV2. Though it does not correspond to any crystal lattice, this kind of symmetry has the advantage of exhibiting in a simple way three different Young moduli along three orthogonal directions: this property suits well with the elastic properties of various amorphous materials as well as some rocks, soils or cracked solids. Note that the composition of several such transformations does not give access to different kinds of anisotropy.

If 2 of the 3 Young moduli (E_1, E_2, E_3) are equal, say $E_1 = E_2$, we are left with a special case of transverse isotropy (denoted TITrI, transverse isotropic transformed isotropy, in the diagram of Fig. 3), depending on 3 parameters only (E_1, E_3, ν) instead of 5 $(E_1, E_3, \nu_{12}, \nu_{13}, \mu_{13})$. In this case, we have $\nu_{12} = \nu$, $\nu_{13} = \nu\sqrt{E_1/E_3}$ (which means $\sqrt{\nu_{13}\nu_{31}} = \nu$) and $\mu_{13} = \sqrt{E_1E_3}/[2(1+\nu)]$; the compliance matrix reads

$$[\mathbf{S}]^{\mathrm{TITrI}} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu}{E_1} & \frac{-\nu}{\sqrt{E_1E_3}} & & & \\ \frac{-\nu}{E_1} & \frac{1}{E_1} & \frac{-\nu}{\sqrt{E_1E_3}} & & & \\ \frac{-\nu}{\sqrt{E_1E_3}} & \frac{-\nu}{\sqrt{E_1E_3}} & \frac{1}{E_3} & & & \\ & & & \frac{2(1+\nu)}{\sqrt{E_1E_3}} & & \\ & & & & \frac{2(1+\nu)}{\sqrt{E_1E_3}} & \\ & & & & & \frac{2(1+\nu)}{E_1} \end{bmatrix} \tag{67}$$

and according to the last equation of (66), the $\boldsymbol{P}$-transformation reduces to

$$[\boldsymbol{P}]^{\mathrm{TITrI}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & a \end{bmatrix}, \quad a = \left(\frac{E_3}{E_1}\right)^{1/4} \tag{68}$$

From the decomposition of any vector $\underline{V}$ into its transverse and axial components, say $\underline{V} = \underline{V}_T + V_3\,\underline{e}_3$, with $\underline{V}_T \cdot \underline{e}_3 = 0$, this transformation operates on $\underline{x}$, $\underline{u}$, and the volume and surface forces $\underline{F}$ and $\underline{T}$ according to the relations

$$\begin{cases} \tilde{\underline{x}}_T = \underline{x}_T, & \tilde{\underline{u}}_T = \underline{u}_T, \quad \tilde{\underline{F}}_T = \underline{F}_T \\ \tilde{x}_3 = \frac{1}{a}x_3, & \tilde{u}_3 = au_3, \quad \tilde{F}_3 = \frac{1}{a}F_3 \\ \tilde{\underline{T}}_T = \kappa\underline{T}_T, & \tilde{\underline{T}}_3 = \frac{\kappa}{a}T_3, \quad \kappa(\underline{n}) = \left(\sqrt{n_1^2 + n_2^2 + a^2n_3^2}\right)^{-1} \end{cases} \tag{69}$$

4.2.2.2. Transformed transverse isotropy. The foregoing analysis can be extended to transverse isotropic transformed moduli $\tilde{\mathbb{C}}$, along the axis $\underline{n}$; let b_i, $i = 1$ to 5, the five corresponding independent parameters:

$$\begin{aligned} \tilde{\boldsymbol{C}}_{ijkl} = {} & b_1\delta_{ij}\delta_{kl} + b_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + b_3(\delta_{ij}n_kn_l + \delta_{kl}n_in_j) \\ & + b_4(\delta_{ik}n_jn_l + \delta_{il}n_kn_j + \delta_{jk}n_in_l + \delta_{jl}n_kn_i) + b_5n_in_jn_kn_l \end{aligned} \tag{70}$$

With the axis $\underline{x}_3$ along $\underline{n}$, the stiffness matrix reads as the following where the values denoted by a star (*) derive from the other values by symmetry relations:

$$[\widetilde{\mathbf{C}}] = \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} & & & \\ \tilde{c}_{12} & \tilde{c}_{11} & \tilde{c}_{13} & & & \\ \tilde{c}_{13} & \tilde{c}_{13} & \tilde{c}_{33} & & & \\ & & & \tilde{c}_{44} & & \\ & & & & \tilde{c}_{44} & \\ & & & & & \tilde{c}^*_{66} \end{bmatrix}, \quad \tilde{c}^*_{66} = \frac{\tilde{c}_{11} - \tilde{c}_{12}}{2} \tag{71}$$

with

$$\begin{cases} \tilde{c}_{11} = b_1 + 2b_2, \quad \tilde{c}_{12} = b_1, \quad \tilde{c}_{13} = b_1 + b_3 \\ \tilde{c}_{33} = b_1 + 2b_2 + 2b_3 + 4b_4 + b_5, \quad \tilde{c}_{44} = b_2 + b_4 \end{cases} \tag{72}$$

With use of (49) and (50), the corresponding general form for $\mathbb{C}$ is found to be

$$\begin{aligned} C_{ijkl} = {} & b_1 D_{ij} D_{kl} + b_2 (D_{ik} D_{jl} + D_{il} D_{jk}) \\ & + b_3 (D_{ij} N_k N_l + D_{kl} N_i N_j) b_4 (D_{ik} N_j N_l + D_{il} N_k N_j + D_{jk} N_i N_l + D_{jl} N_k N_i) + b_5 N_i N_j N_k N_l \end{aligned} \tag{73}$$

with $\boldsymbol{D} = \boldsymbol{P} \cdot \boldsymbol{P}^{\mathrm{T}}$ and $\underline{N} = \boldsymbol{P} \cdot \underline{n}$. For given eigendirections of $\boldsymbol{D}$, we can choose 3 eigenvalues for $\boldsymbol{D}$, 2 angles for the direction $\underline{n}$ and 4 independent parameters b_2 to b_5 (b_1 can be chosen with $\boldsymbol{D}$). This shows that $\mathbb{C}$ depends on 9 independent parameters (plus 3 additional parameters for possible rotations of eigendirections of $\boldsymbol{D}$). This class of symmetry is denoted TrTI (transformed transverse isotropy) in the diagram of Fig. 3 below. According to our approach, it could be considered as an extension of Saint-Venant's anisotropy, TrI.

Special attention is paid now to the case when $\underline{n}$ is an eigendirection for $\boldsymbol{D}$. Let ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$), with $\underline{e}_3 = \underline{n}$, be an eigenbasis of $\boldsymbol{D}$ and d_α ($\alpha = 1, 2, 3$) its eigenvalues. The $\boldsymbol{P}$ tensor is diagonal in this basis with the eigenvalues $a_\alpha = \sqrt{d_\alpha}$. As long as the class of symmetry of $\mathbb{C}$ only, which is not modified by multiplication by a scalar, is concerned, $\boldsymbol{P}$ can be chosen such that $|\boldsymbol{P}| = 1$. In addition, any stretching along $\underline{e}_3$ does not change the property of transverse isotropy around $\underline{e}_3$ so that a_3 can be fixed as 1 without restriction. The eigenvalues of $\boldsymbol{P}$ are then $a_1 = a$, $a_2 = a^{-1}$ and 1. The associated $\mathbb{C}$ tensor is then orthotropic with a 6×6 matrix which reads

$$[\boldsymbol{C}]^{\mathrm{STrTI}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c^*_{23} & & & \\ c_{13} & c^*_{23} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c^*_{55} & \\ & & & & & c^*_{66} \end{bmatrix}, \quad \begin{aligned} c^*_{23} &= \sqrt{\frac{c_{22}}{c_{11}}} c_{13} \\ c^*_{55} &= \sqrt{\frac{c_{11}}{c_{22}}} c_{44} \\ c^*_{66} &= \frac{\sqrt{c_{11} c_{22}} - c_{12}}{2} \end{aligned} \tag{74}$$

with

$$c_{11} = a^4 \tilde{c}_{11}, \quad c_{22} = a^{-4} \tilde{c}_{11}, \quad c_{33} = \tilde{c}_{33}, \quad c_{12} = \tilde{c}_{12}, \quad c_{13} = a^2 \tilde{c}_{13}, \quad c_{44} = a^{-2} \tilde{c}_{44} \tag{75}$$

This matrix depends on 6 independent parameters (c_{11}, c_{22}, c_{33}, c_{12}, c_{13}, c_{44}). The corresponding model is named STrTI (symmetric transformed transverse isotropy) in the diagram of Fig. 3.

Conversely, if the stiffness matrix has the form (74) in the basis ($\underline{e}_1, \underline{e}_2, \underline{e}_3$), $\boldsymbol{P}$ can be chosen diagonal in this basis with the eigenvalues $a_1 = a$, $a_2 = a^{-1}$, $a_3 = 1$ and

$$a = \left(\frac{c_{11}}{c_{22}}\right)^{\frac{1}{8}} \tag{76}$$

The $\boldsymbol{P}$-transformed tensor of $\mathbb{C}$, say $\widetilde{\mathbb{C}}$, then exhibits transverse isotropy around $\underline{e}_3$ and its stiffness matrix reads as (71) with the relations

$$\tilde{c}_{11} = \sqrt{c_{11}c_{22}}, \quad \tilde{c}_{33} = c_{33}, \quad \tilde{c}_{12} = c_{12}, \quad \tilde{c}_{13} = \sqrt{c_{13}c_{23}}, \quad \tilde{c}_{44} = \sqrt{c_{44}c_{55}} \tag{77}$$

4.2.2.3. Approximation of real elastic materials. The above defined models can be used as good enough approximations for various materials, including crystalline ones, and then yield convenient approximate analytical solutions for some problems such as the search of Green functions (see below). In view of these applications, we discuss now approximations which can be performed for a number of crystals. The elastic stiffness of such crystals are reported in Table 1, as well as the approximations performed according to some of the above proposed models (namely models TrI and STrTI, as well as the reference of transverse isotropy). The reported approximations $\mathbb{C}^{\text{Appr}}$ of the measured moduli $\mathbb{C}^{\text{M}}$ have been obtained by minimization of the distance $\|\mathbb{C}^{\text{Appr}} - \mathbb{C}^{\text{M}}\|$. The last row of the table indicates the relative error, namely $\|\mathbb{C}^{\text{Appr}} - \mathbb{C}^{\text{M}}\|/\|\mathbb{C}^{\text{M}}\|$. The first 3 crystals are orthorhombic, with 9 independent parameters, the three others are transversely isotropic with 5 independent parameters. Of course, the approximation obtained with the STrTI model (6 parameters) is always better than the one deriving from the approximation of transverse isotropy (5 parameters).

Table 1
Elastic stiffnesses for a set of crystals (Dieulesaint and Royer, 1974) and their approximation by several anisotropic models

	$C_{\alpha\alpha}$ (10^{10} N/m^2)									Err. (%)
	c_{11}	c_{22}	c_{33}	c_{44}	c_{55}	c_{66}	c_{12}	c_{13}	c_{23}	
$KB_5O_8 \cdot 4H_2O$	5.82	3.59	2.55	1.64	0.463	0.57	2.29	1.74	2.31	–
Model TrI	5.346	3.818	2.806	0.738	0.874	1.019	2.479	2.125	1.796	.25289
Trans. isotropy	4.388	4.388	2.550	1.052	1.052	0.889	2.609	1.025	1.025	.26167
Model STrTI	5.112	3.794	2.550	0.933	1.083	0.895	2.615	2.141	1.844	.24195
S	2.40	2.05	4.83	0.43	.87	.76	1.33	1.71	1.59	–
Model TrI	2.519	2.011	4.710	0.725	0.811	.530	1.190	1.822	1.627	.11158
Trans. isotropy	2.381	2.381	4.830	0.650	0.650	.604	1.174	1.650	1.650	.11059
Model STrTI	2.619	2.134	4.830	0.626	0.693	0.598	1.168	1.733	1.564	.09572
$BaSO_4$	8.8	7.81	10.4	1.17	2.79	2.55	4.77	2.69	2.89	–
Model TrI	8.975	7.446	9.294	2.374	2.606	2.333	3.509	3.921	3.571	.19480
Trans. isotropy	8.696	8.696	10.400	1.979	1.979	2.159	4.379	2.790	2.790	.13472
Model STrTI	9.419	7.957	10.400	1.919	2.088	2.146	4.366	2.898	2.664	.12249
$BaTiO_3$	16.8	16.8	18.9	5.46	5.46	4.49	7.82	7.10	7.10	–
Model TrI	16.828	16.828	18.920	5.206	5.206	4.910	7.008	7.431	7.431	.04364
Zn	16.1	16.1	6.10	3.83	3.83	6.34	3.42	5.01	5.01	–
Model TrI	16.407	16.407	7.340	3.815	3.815	5.704	4.999	3.343	3.343	.14248
Co	30.7	30.7	35.81	7.53	7.53	7.10	16.5	10.30	10.30	–
Model TrI	29.322	29.322	33.281	8.750	8.750	8.213	12.896	13.739	1.739	.14471

For each crystal, the measured moduli $\mathbb{C}^{\text{M}}$ according to the quoted reference are reported on the first line whereas different approximations $\mathbb{C}^{\text{Appr}}$ are reported on the other lines with the associated relative error ($\|\mathbb{C}^{\text{Appr}} - \mathbb{C}^{\text{M}}\|/\|\mathbb{C}^{\text{M}}\|$) indicated in the last row.

It is interesting to note that, for some cases (e.g. for Potassium pentaborate $KB_5O_8 \cdot 4H_2O$), the error attached to the TrI model (4 parameters) is lower than that coming from use of a transverse isotropic approximation (5 parameters). For some transverse isotropic materials (e.g. for Barium titanate $BaTiO_3$), the TrI model yields a very low error (≈4%). Taking account of the experimental uncertainty, it can be more convenient to use for it the Green function derived for the TrI model (see below) which looks simpler than the one attached to transverse isotropic.

5. New analytical derivation of Green functions

In this section, we give closed form expressions of the Green tensor for an infinite or semi-infinite medium for the TrI model (transformed from isotropy, see (62) or (64)).

5.1. Infinite medium

The elastic moduli $\mathbb{C}$ are given by (61) with $\boldsymbol{D} = \boldsymbol{P} \cdot \boldsymbol{P}^{\mathrm{T}}$ symmetric and positive definite. The $\boldsymbol{P}$-transformed moduli $\tilde{\mathbb{C}}$ are isotropic, with Lamé coefficients λ and μ. The Green–Kelvin tensor for an infinite body with such moduli is known to be

$$\widetilde{G}_{ij}(\underline{\tilde{x}} - \underline{\tilde{x}}') = \frac{1}{4\pi\mu}\left\{\frac{\delta_{ij}}{\|\underline{\tilde{x}} - \underline{\tilde{x}}'\|} - \frac{1}{4(1-v)}\tilde{\partial}_{ij}\|\underline{\tilde{x}} - \underline{\tilde{x}}'\|\right\} \tag{78}$$

with $v = \lambda/[2(\lambda + \mu)]$ and $\tilde{\partial}_i(\cdot) = \frac{\partial}{\partial \tilde{x}_i}(\cdot)$. By replacing in (78) $\underline{\tilde{x}}$, $\underline{\tilde{x}}'$ and $\widetilde{\boldsymbol{G}}$ by their expression from (49) and (58), with $\partial_i(\cdot) = R_{ij}\tilde{\partial}_j(\cdot)$, $\boldsymbol{R} = (\boldsymbol{P}^{\mathrm{T}})^{-1}$, $\tilde{\partial}_i(\cdot) = P_{ji}\partial_i(\cdot)$ and $\rho(\underline{x}) = \sqrt{\underline{x} \cdot \boldsymbol{H} \cdot \underline{x}}$, $\boldsymbol{H} = \boldsymbol{D}^{-1}$, we find (Pouya, 2000)

$$G_{ij}(\underline{x} - \underline{x}') = \frac{\sqrt{|\boldsymbol{H}|}}{4\pi\mu}\left\{\frac{H_{ij}}{\rho(\underline{x} - \underline{x}')} - \frac{1}{4(1-v)}\partial_{ij}\rho(\underline{x} - \underline{x}')\right\} \tag{79}$$

If $\boldsymbol{D}$ has two identical eigenvalues (TITrI model (67)), a special case of transverse isotropy is obtained. Eq. (79) specified for this case is consistent with the general expression of the Green tensor for transverse isotropy given by Pan and Chou (1976), except for a mistake in this expression: the second member of Eq. (18) in Pan and Chou (1976) must be multiplied by $(c_{11}/c_{33})^{1/4}$.

5.2. Semi-infinite body

The case of isotropy has been solved by Mindlin (1936). We could use this solution for an extension, thanks to (58), to moduli $\mathbb{C}$ given by (61). Nevertheless, Mindlin's solution is usually given (Mindlin, 1936; Mura, 1982; Bonnet, 1995) in a specific basis, with one direction normal to the plane boundary, whereas, since we are interested with anisotropic elasticity, we need an intrinsic general expression. It can be obtained with help of the following notation:

$$\xi = \underline{x} \cdot \underline{n}, \quad \xi' = \underline{x}' \cdot \underline{n}, \quad \underline{X} = \underline{x} - \underline{x}', \quad R_1 = \|\underline{X}\|, \quad R_2 = (R_1^2 + 4\xi\xi')^{1/2} \tag{80}$$

with $\underline{n}$ the unit outward normal to the isotropic half-space with Poisson's ratio v. Then, according to Mura (1982) , Mindlin's solution reads

$$\boldsymbol{G}(\underline{x}, \underline{x}') = \frac{1}{16\pi\mu(1-v)}[A_1\boldsymbol{I} + A_2\underline{X} \otimes \underline{X} + A_3\underline{X} \otimes \underline{n} + A_4\underline{n} \otimes \underline{X} + A_5\underline{n} \otimes \underline{n}] \tag{81}$$

with $\boldsymbol{I}$ the second-order unit tensor and the definitions

$$\begin{cases} A_1 = \dfrac{3-4v}{R_1} + \dfrac{1}{R_2} + \dfrac{R_2^2 - R_1^2}{2R_2^3} + \dfrac{4(1-v)(1-2v)}{R_2+\xi+\xi'} \\ A_2 = \dfrac{3-4v}{R_2^3} + \dfrac{1}{R_1^3} - \dfrac{6\xi\xi'}{R_2^5} - \dfrac{4(1-v)(1-2v)}{R_2(R_2+\xi+\xi')^2} \\ A_3 = \dfrac{12\xi^2\xi'}{R_2^5} - \dfrac{4(1-v)(1-2v)}{R_2(R_2+\xi+\xi')^2}(R_2+2\xi') \\ A_4 = -\dfrac{12\xi\xi'^2}{R_2^5} + \dfrac{4(1-v)(1-2v)}{R_2(R_2+\xi+\xi')^2}(R_2+2\xi) \\ A_5 = \dfrac{4(1-v)(1-2v)}{R_2(R_2+\xi+\xi')^2}[R_2(\xi+\xi')+4\xi\xi'] + \dfrac{8\xi\xi'(1-2v)}{R_2^3} + \dfrac{24\xi^2\xi'^2}{R_2^5} \end{cases} \tag{82}$$

Let now the anisotropic half-space with the moduli (61) of the TrI model be defined by $\underline{x}\cdot\underline{n} \geqslant 0$. It is transformed by the tensor $\boldsymbol{P}$ such that $\boldsymbol{P}\cdot\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{D}$ into an isotropic half-space with the normal $\tilde{\underline{n}}$ transformed from $\underline{n}$ by $\tilde{\underline{n}} = \|\boldsymbol{P}^{\mathrm{T}}\cdot\underline{n}\|^{-1}\boldsymbol{P}^{\mathrm{T}}\cdot\underline{n}$. Note that $\boldsymbol{P}$ can be chosen symmetric. The associated Green tensor $\widetilde{\boldsymbol{G}}$ is then given by formulae (80)–(82) which depend on $\tilde{\underline{x}}$, $\tilde{\underline{x}}'$ and $\tilde{\underline{n}}$. By use of (58) where $\tilde{\underline{x}}$, $\tilde{\underline{x}}'$ and $\tilde{\underline{n}}$ are known functions of $\underline{x}$, $\underline{x}'$ and $\underline{n}$, the expected Green tensor $\boldsymbol{G}$ is obtained. It reads

$$\boldsymbol{G}(\underline{x},\underline{x}') = \frac{\sqrt{|\boldsymbol{H}|}}{16\pi\mu(1-v)}[B_1\boldsymbol{H} + B_2\boldsymbol{H}\cdot(\underline{X}\otimes\underline{X})\cdot\boldsymbol{H} + \kappa B_3\boldsymbol{H}\cdot(\underline{X}\otimes\underline{n}) + \kappa B_4(\underline{n}\otimes\underline{X})\cdot\boldsymbol{H} + \kappa^2 B_5(\underline{n}\otimes\underline{n})] \tag{83}$$

where we have put

$$\begin{cases} v = \lambda/[2(\lambda+\mu)], \quad \kappa = (\underline{n}\cdot\boldsymbol{D}\cdot\underline{n})^{-1/2} \\ R_1^* = \sqrt{\underline{X}\cdot\boldsymbol{H}\cdot\underline{X}}, \quad \xi^* = \kappa\underline{x}\cdot\underline{n}, \quad \xi'^* = \kappa\underline{x}'\cdot\underline{n}, \quad R_2^* = (R_1^{*2} + 4\xi^*\xi'^*)^{1/2} \end{cases} \tag{84}$$

with $\boldsymbol{H} = \boldsymbol{D}^{-1}$, and where the parameters B_i are defined from $(R_1^*, R_2^*, \xi^*, \xi'^*, v)$ in the same way as A_i were defined from (R_1, R_2, ξ, ξ', v) in (82).

Now again, we can focus on the particular case of the TITrI model (67) for which the symmetry axis is normal to the plane boundary: this situation frequently occurs in soil mechanics for foundations problems with different vertical and horizontal moduli for the soil. According to experimental data given by Boehler (1975) , the TITrI model is well suited to the elastic anisotropy of various soils (Pouya and Reiffsteck, 2003). The proposed transformation then reduces to the simple Eqs. (69) and can be conveniently used for extending known solutions of foundation problems for isotropic soils to media exhibiting this kind of anisotropy. These problems usually make recourse to Boussinesq's (1885) solution which is concerned with the special case of Mindlin's solution for a point force $W\underline{e}_3$ applied on the plane boundary in the normal direction, where $\underline{e}_3$ is also a symmetry axis for the anisotropy defined by (E_1, E_3, v) according to (67). With $a = (E_3/E_1)^{1/4}$, the transformation defined by (69), (68) and (52) of the classical Boussinesq's solution (Johnson, 1992) yields the solution of the Boussinesq problem for the considered anisotropy. Note that W is transformed as F_3 in (69), namely $\tilde{W} = W/a$. In cylindrical coordinates of axis $\underline{e}_3$ (see Fig. 2a), and with the notation

$$R = \sqrt{a^2r^2 + z^2}$$

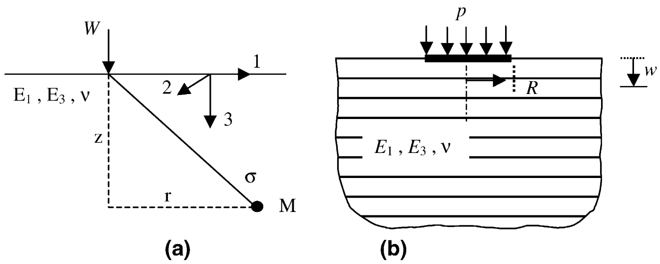

Fig. 2. (a) Point force on an infinite soil for the TITrI model; (b) rigid circular footing on an infinite soil for the TITrI model.

this solution reads

$$\begin{cases} u_z = \dfrac{W}{4\pi\mu a}\left[\dfrac{2(1-\nu)}{R} + \dfrac{z^2}{R^3}\right], \quad u_r = \dfrac{Wa}{4\pi\mu}\left[-\dfrac{(1-2\nu)r}{R(R+z)} + \dfrac{rz}{R^3}\right] \\ \sigma_{zz} = \dfrac{-3W}{2\pi}a^3\dfrac{z^3}{R^5}, \quad \sigma_{rr} = \dfrac{Wa}{2\pi}\left[\dfrac{1-2\nu}{R(R+z)} - \dfrac{3z}{R^3}\left(1-\dfrac{z^2}{R^2}\right)\right] \\ \sigma_{rz} = \dfrac{-3W}{2\pi}a^3\dfrac{rz^2}{R^5}, \quad \sigma_{\theta\theta} = \dfrac{W(1-2\nu)a}{2\pi}\left[\dfrac{z}{R^3} - \dfrac{1}{R(R+z)}\right] \end{cases} \tag{85}$$

The displacement field for a number of foundation problems in anisotropic soils with the properties (67) can be easily derived from this basic solution. In addition, many results can be derived from the direct transformation of known results for isotropic soils. For instance, the vertical displacement w of a rigid circular footing (Fig. 2b) for a half-space with the elastic properties (67) is derived directly through (69) from the classical Boussinesq result for isotropic soils, namely

$$\tilde{w} = \frac{\pi}{2}\frac{(1-\nu^2)}{E}\tilde{p}\tilde{R} \tag{86}$$

and reads

$$w = \left(\frac{E_3}{E_1}\right)^{1/4}\frac{\pi}{2}\frac{(1-\nu^2)}{E_3}pR \tag{87}$$

with $E = E_1$: this is due to the fact that the transformation does not change the geometry of the problem (half-space and circular footing with invariant radius) and leads to $\tilde{w} = aw$, $\kappa = 1/a$ and $\tilde{p} = p/a^2$. Note that a number of other similar problems, e.g. the solution for a trapezoidal embankment (Gray, 1936), which derive from the basic Boussinesq solution, can be generalized in this way too.

5.3. Further extensions

All the above mentioned models of elastic anisotropy are reported in the diagram of Fig. 3 where each model located at some level is a generalization of the models situated beneath. Ellipses bordered with a thin line indicate cases for which the Green functions are already known whereas a thick border correspond to cases for which it has been or can be derived through the proposed transformation. Dashed lines indicate cases which cannot be correlated with those at a lower level by a linear transformation so that the Green function cannot be derived in the same way.

Note that the number of parameters indicated on the diagram of Fig. 3 does not integrate the 3 rotational degrees of freedom (2 for axial symmetry cases) which should be added to the former (e.g. 12

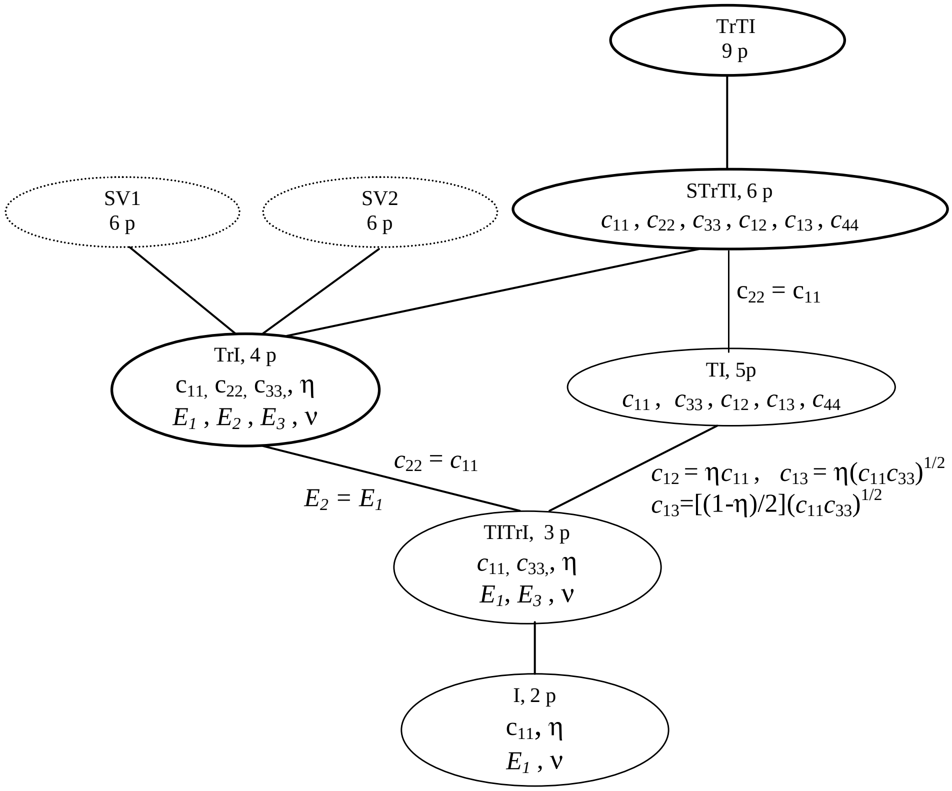

Fig. 3. Diagram showing different anisotropic models and their mutual relations (SV1: $\sqrt[4]{E(\underline{n})}$ is ellipsoidal; SV2: $\sqrt[-4]{c(\underline{n})}$ is ellipsoidal; TrI: transformed isotropy, SV1 ∩ SV2; I: isotropy; TI: transverse isotropy; TrTI: transformed TI; STrTI: symmetric transformed TI; TITrI: transverse isotropic transformed isotropy, TrI ∩ TI).

parameters instead of 9 for the TrTI model) in view of comparison with the maximum 21 parameters when no symmetry is present.

6. Other extensions and conclusion

In view of illustrating the potential applicability of the proposed transformation, we have concentrated on problems for which the geometry is invariant (point force in an infinite) or belongs to an invariant geometry family (semi-infinite body). Another interesting domain of application is concerned with the general solutions of the equilibrium equations for elasticity such as the solutions for stress or strain potentials for which no geometrical aspect is present (Lamé's potential or Galerkine's, Papkovich's (1932) and Neuber's (1934) solutions for isotropy, etc.).

Besides the ellipsoidal inclusion problem in an infinite medium, several isotropic solutions can be extended to anisotropy in a similar way, such as the point force in a layered medium (Benitez and Rosakis's, 1987)or in one of the two joined semi-infinite solids (Rongved's, 1955), or as the torsion of a cone (Timoshenko, 1934; Lekhnitskii, 1963, p. 341) or the application of an axial compressive force on the apex of a cone (Lekhnitskii, 1963, p. 383), etc. Nevertheless, for the latter case, in order to save the axial symmetry, the transformation must be restricted to an affinity along the cone axis and then to anisotropy of the form (67). In addition to foundation problems for soil mechanics which have been illustrated above, we can also mention that the calculation of stress intensity factors for cracks in anisotropic media can be performed with the same method (work in progress).

Two more conclusions can be drawn in view of further advances:

- we have mainly developed a particular simple solution of the general equations for the transformation of an elastic boundary value problem into another one, namely linear transformations of the coordinates and displacements. Many other applications are likely to be open for more general transformations, including inhomogeneous ones. These possibilities still remain to be explored;
- the general form (9) and (10) of the considered transformations is a local one. Even in the restricted framework of linear transformations, nonlocal (e.g. integral) formulations could have been considered as well: this would have probably enlarged the finding of admissible transformations.

Appendix A. Relation between $\boldsymbol{M}$ and $\boldsymbol{Q}$

This appendix refers to the end of Section 3, related to the connection which can exist between tensors $\boldsymbol{M}$ and $\boldsymbol{Q}$. Let us first notice that for every $\mathbb{C}$, the tensor $\overline{\mathbb{C}}$ defined by (47) has the symmetries $\overline{C}_{ikjl} = \overline{C}_{iklj} = \overline{C}_{kijl} = \overline{C}_{jlik}$, but is not always *definite*, i.e., it does not satisfy (48) necessarily. For instance, if $\mathbb{C}$ is defined by $c_{11} = c_{22} = c_{33} = c_{44} = c_{55} = c_{66} = c > 0$, with all the other coefficients equal to zero, it is easy to check that $\overline{\mathbb{C}}$ is not definite. When $\mathbb{C}$ is isotropic, $\overline{\mathbb{C}}$ is definite: as a matter of fact, if λ and μ are the Lamé constants of $\mathbb{C}$, then $\overline{\mathbb{C}}$ is isotropic with Lamé constants $\bar{\lambda} = \mu$ and $\bar{\mu} = (\lambda + \mu)/2$, and the inequalities $\mu > 0$ and $3\lambda + 2\mu > 0$ which express that $\mathbb{C}$ is definite imply also that $\bar{\mu} > 0$ and $3\bar{\lambda} + 2\bar{\mu} > 0$.

Let us now suppose that $\mathbb{C}$ is such that $\overline{\mathbb{C}}$ is *definite*. The right side of (42) is antisymmetric with respect to (i, k) and the same property must hold for the left side; so:

$$\overline{C}_{ikjl}[Q_{jm}M_{\ln} - Q_{\ln}M_{jm}] = 0 \tag{A.1}$$

Since $\overline{\mathbb{C}}$ is definite, (A.1) implies

$$Q_{jm}M_{\ln} - Q_{\ln}M_{jm} + Q_{lm}M_{jn} - Q_{jn}M_{lm} = 0 \tag{A.2}$$

Multiplying both sides of (A.2) by $(\boldsymbol{Q}^{-1})_{nj}$ leads to

$$M_{lm} = (1/3)(\boldsymbol{M}\boldsymbol{Q}^{-1})_{jj}Q_{lm} \tag{A.3}$$

This means that $\boldsymbol{M}$ is proportional to $\boldsymbol{Q}$. So we can write

$$\boldsymbol{M}(\underline{x}) = q(\underline{x})\boldsymbol{Q}(\underline{x}) \tag{A.4}$$

Substitution of this expression of $\boldsymbol{M}$ in (42) shows that the left side of (42) vanishes and then that $\underline{\Psi}^{(mn)}$ must be antisymmetric with respect to (m, n). The same property holds for its rotational. According to (37) and (22):

$$\nabla \wedge \underline{\Psi}(mn) = \underline{\chi}^{(mn)} = \boldsymbol{\Sigma}^{(m)} \cdot \underline{\xi}'^{(n)} - \boldsymbol{\Sigma}'^{(n)} \cdot \underline{\xi}^{(m)} \tag{A.5}$$

Substitution of (A.4) in (20) and (21) leads to:

$$\underline{\xi}'^{(n)} = q\underline{\xi}^{(n)}, \quad \boldsymbol{\Sigma}'^{(n)} = q\boldsymbol{\Sigma}^{(n)} + \mathbb{C} : (\nabla q \otimes \underline{\xi}^{(n)}) \tag{A.6}$$

so that (A.5) reads

$$\nabla \wedge \underline{\Psi}^{(mn)} = \{q(\boldsymbol{\Sigma}^{(m)} \cdot \underline{\xi}^{(n)} - \boldsymbol{\Sigma}^{(n)} \cdot \underline{\xi}^{(m)})\} - \{[\mathbb{C} : (\nabla q \otimes \underline{\xi}^{(n)})] \cdot \underline{\xi}^{(m)}\} \tag{A.7}$$

In (A.7), the first term between { } is antisymmetric in (m, n). For the whole expression to be antisymmetric, the second term between { } must be antisymmetric too. So

$$[\mathbb{C} : (\nabla q \otimes \underline{\xi}^{(n)})] \cdot \underline{\xi}^{(m)} + [\mathbb{C} : (\nabla q \otimes \underline{\xi}^{(m)})] \cdot \underline{\xi}^{(n)} = 0 \tag{A.8}$$

or, equivalently

$$C_{ijkl}\,\partial_k q(Q_{jm}Q_{ln} + Q_{jn}Q_{lm}) = 0 \tag{A.9}$$

Multiplication of both sides by $(\boldsymbol{Q}^{-1})_{mp}(\boldsymbol{Q}^{-1})_{np}$ leads to

$$\boldsymbol{\Gamma} \cdot \nabla q = 0 \tag{A.10}$$

with $\Gamma_{ik} = C_{ipkp}$. Since $\boldsymbol{\Gamma}$ is well known to be symmetric and positive definite, (A.10) implies $\nabla q = 0$, i.e., q constant and so: $\forall \underline{x} \in \Omega$, $\boldsymbol{M}(\underline{x}) = q\boldsymbol{Q}(\underline{x})$.

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